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G.Sh. Poltyryev

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SOME PROPERTIES OF HADAMARD MATRICES OF ORDER $N = 2^m$ AND THE REDUCTION OF SIGNAL REDUNDANCY BY USE OF DIGITAL FILTERS

G.Sh. Poltyryev

ABSTRACT: An examination is made of the decomposition of a random sequence formed by discretization of a continuous stationary random process x(t) in terms of column vectors of a normalized Hadamard matrix. A device executing this decomposition is called an Hadamard filter. A number of properties of Hadamard matrices of an order $N=2^m$, for m = 1, 2, ..., are proven. On the basis of these properties, certain properties of the variances of the outputs of the filter have been established. It is shown that the relative number of data at the output of the filter that ensure the necessary mean square error is independent of its dimensions for processes with a linear correlation function and depend only slightly on them for processes with a near-linear correlation function. It is also shown that, for processes with correlation function close to quadratic, the ratio of the number n of data at the output of the filter to the dimension of the filter N is, under certain conditions, $n/N = (1 + \log N)/N$. /518*

For the transmission of a continuous signal along a discrete channel of communication, three operations are necessary: discretization, quantization, and coding. Obviously, the first two operations cause the transmitted signal to differ from the original one. In the present article, we take as criterion of the difference between the transmitted and the received signals the mean square of their difference—the variance of the error. /519

Suppose that a continuous random process $\dot{x}(t)$ is discretized by taking readings of it with some constant interval Δt. Here, we assume that Δt is chosen sufficiently small that the error of renewal of the process between the readings is

*Numbers in the margin indicate pagination in the foreign text.

much less than the maximum admissible error. In [1], an examination is made of a method of transforming the vector of the readings $X = [x_1, x_2, \ldots, x_N]$ into a vector $Y = [y_1, \ldots, y_n]$, where $n \leq N$, with uncorrelated components by means of a linear orthogonal operator. The device executing this transformation was called a digital filter in [1]. It is easy to see that, in such filters, the random sequence will be decomposed on the interval $T = N\Delta t$ into eigenvectors of the correlation matrix. We know [2] that such a decomposition is optimal in the sense of minimizing the variance of the error. However, its practical application is severely limited by the complexity of realization.

In the present article, we shall consider digital filters that execute the decomposition of a random sequence into a system of orthogonal vectors, forming a normalized Hadamard matrix (Hadamard filters).

1. Suppose that $A = [a_{ij}]$ is an orthonormal square matrix of order N. We denote by A_n the $N \times n$ matrix whose columns are n of the columns of A. Let us define a vector $\hat{X} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_N]$ as follows:

$$\hat{X} = X A_n A_n' \tag{1}$$ /520

where A_n' denotes the transpose of A_n.

One can easily show that

$$\delta^2_{[n]} = \frac{1}{N} E\{(X-\hat{X})(X-\hat{X})'\} = \frac{1}{N}\sum_{i=1}^{N}\sigma^2_{x_i} - \frac{1}{N}\sum_{i\in[n]}\sigma^2_{y_i} = \frac{1}{N}\sum_{i\in[N]\setminus[n]}\sigma^2_{y_i} \tag{2}$$

where [N] is the set of indices of the columns of the matrix A,

[n] is the set of indices of the columns of the matrix A_n,

$$\sigma^2_{x_i} = E\{x_i^2\}$$

$$\sigma^2_{y_i} = E\{y_i^2\}$$

$\sigma^2_{y_i} = E[y_i^2]$, y_i is the coordinate of the vector,

$$Y = XA \tag{3}$$

Thus, if X is the vector of readings of the random process x(t) and $\hat{X}$ is its estimate with respect to the vector $Y_n = X \cdot A_n$, it follows from (2) that the mean square error in the reading is equal to the sum of the variances of the coordinates of the vector Y that do not appear in the vector Y_n.

Suppose that the matrix A is a normalized Hadamard matrix or order N. The elements of such a matrix are equal to $\pm 1/\sqrt{N}$. Suppose that $R = [r_{ij}]$, for $i, j = 1, \ldots, N$, is the correlation matrix of the vector X. We shall consider the process x(t) stationary with unit variance and zero mathematical expectation. Let us consider the expression for the variance of the ith coordinate of the vector Y:

$$\sigma_{y_i}^2 = E\{y_i^2\} = E\{(XA_i)(XA_i)'\} = E\{A_i'X'XA_i\} = \tag{4}$$
$$= A_i' R A_i = \sum_{k=1}^{N}\sum_{l=1}^{N} a_{ki} r_{kl} a_{li}$$

where A_i is the ith column of the matrix A.

Remembering that, for a stationary process, $r_{kl} = r_{(k-l)} = r(\tau)$; $r_{kk} = 1$, $\sum_{k=1}^{N} a_{ki}^2 = 1$, we obtain

$$\sigma_{y_i}^2 = 1 + \frac{2}{N}\sum_{\tau=1}^{N-1} r(\tau) \sum_{k=1}^{N} h(k,i)\, h(k+\tau,i) = \tag{5}$$
$$= 1 + \frac{2}{N}\sum_{\tau=1}^{N-1} r(\tau)\, H(\tau,i,N)$$

where h(k, i) = $\pm$ 1 for $k \leq N$ and h(k, i) = 0 for $k > N$ /521

$$H(\tau,i,N) = \sum_{k=1}^{N-1} h(k,i)\, h(k+\tau,i) \tag{6}$$

2. Let us look at several properties of Hadamard matrices associated with the function $H(\tau,i,N)$.

Lemma 1. Let h(i, j) = $\pm$ 1 denote an element of an Hadamard matrix and suppose that h(i, j) $\equiv$ 1. Then,

$$\sum_{\tau=1}^{N-1} H(\tau,i,N) = \begin{cases} -\frac{N}{2} & i \neq 1 \\ \frac{N(N-1)}{2} & i = 1 \end{cases} \tag{7}$$

Let us not consider Hadamard matrices of order $N = 2^m$, for m = 1, 2, ... A matrix of order N/2 can be obtained from an Hadamard matrix of order N/2 if we replace each element of the latter with the matrix

$$\begin{Bmatrix} 1 & 1 \\ 1 & -1 \end{Bmatrix} \tag{8}$$

For our subsequent constructions, it will be convenient to consider Hadamard matrices of order $N = 2^m$ constructed in the following manner: The first N/2 columns of the matrix of order N are obtained from the corresponding column of a matrix of order N/2 by successive repetition of each of its elements. The odd elements of the (N/2 + i)th column coincide with the elements of the ith column of the matrix of order N/2, and the even elements constitute their inverse. We shall refer to matrices constructed in this manner as canonical Hadamard matrices of order $N = 2^m$. One can easily see that Hadamard matrices of order $N = 2^m$ formed by iteration of the matrix (8) differ from canonical matrices only in the order of their columns.

Lemma 2. Suppose that (h_{ij}) is a canonical Hadamard matrix of order $N = 2^m$. Then,

$$H(\tau,i,N) = \begin{cases} H(\lfloor\frac{\tau}{2}\rfloor, i, \frac{N}{2}) + H(\lceil\frac{\tau}{2}\rceil, i, \frac{N}{2}) & i \leq \frac{N}{2} \\ (-1)^{\tau}\left(H(\lfloor\frac{\tau}{2}\rfloor, i-\frac{N}{2}, \frac{N}{2}) + H(\lceil\frac{\tau}{2}\rceil, i-\frac{N}{2}, \frac{N}{2})\right) & i > \frac{N}{2} \end{cases} \tag{9}$$ /522

Lemma 3. Suppose that $H = (h_{i,j})$ is a canonical Hadamard matrix of order $N = 2^m$. Define

$$S(t,i,N) = \sum_{\tau=1}^{N-1} \tau^t H(\tau,i,N) \tag{10}$$

Then,

$$S(1,i,N) = -\frac{2^{2b_i-2}+1}{6} N \quad i \neq 1 \tag{11}$$

$$S(1,1,N) = \frac{N(N^2-1)}{6} \tag{12}$$

$$S(2,i,N) = \begin{cases} 0 & i - \frac{N}{2^{b_i}} \neq 1 \\ -N^2 2^{2(b_i-2)} & i - \frac{N}{2^{b_i}} = 1 \end{cases} \quad i \neq 1 \tag{13}$$

$$S(2,1,N) = \frac{N^2(N^2-1)}{12} \tag{14}$$

where b_i is the smallest integer staisfying the inequality

$$i - \frac{N}{2^{b_i}} > 0 \tag{15}$$

Lemma 4. Suppose that H_N is a canonical Hadamard matrix of order $N=2^m$ and suppose that T_{nl} is an $N \times N/2^l$ matrix whose columns are $N/2^l$ columns of H_N. Suppose that T_i is a column of the matrix T_{Nl}. If $i = \frac{kN}{2^l}+1, \frac{(k+1)N}{2^l}$ for $k = 0, \ldots, 2^{l_i}$, then the matrix $W = T_{Nl} \cdot T'_{Nl}$ is quasidiagonal with equal diagonal blocks of order 2^l.

Lemma 5. Suppose that H_{Nn} is a truncated $n \times n$ canonical Hadamard matrix and /523

$$H_{Nn} = \{T_{Nl_1}, T_{Nl_2}, \ldots, T_{Nl_s}\} \quad \sum_{i=1}^{s} l_i = n \tag{17}$$

where T_{nl_i} is a matrix satisfying the conditions of lemma 4. Then,

1) The matrix $W_n = H_{Nn} H'_{Nn}$ is quasidiagonal with equal diagonal blocks of order 2^{l_v}, where

$$l_v = \max_i l_i \tag{18}$$

2) Each block is

$$W_{n_1} = \frac{N}{2^{l_v}} H_{N,n_1} H'_{N,n_1} \tag{19}$$

where

$$N_1 = 2^{l_\nu} \tag{20}$$

$$H_{N_1 n_1} = \{T_{N,l_1}, T_{N,l_2}, \ldots, T_{N,l_s}\} \tag{21}$$

3. By virtue of these properties of Hadamard matrices, we can establish certain properties of the variances of the components of the vector Y defined by (4):

Theorem 1. Suppose that A_{Nn} is a truncated normalized canonical N x n Hadamard matrix, where $N = 2^m$, that satisfies the conditions of Lemma 5. Then, for the mean square error in the reading σ^2_{Nn}, we have

$$\sigma^2_{Nn} = \sigma^2_{N_1 n_1} \tag{22}$$

where N_1 and n_1 are defined by (20) and (21).

Corollary. If $Y_N = (y_1, y_2, \ldots, y_n)$ is $Y_N = X_N A_N$, where A_N is a normalized canonical Hadamard matrix of order $N = 2^m$, then /524

$$\sum_{\{i\}} \sigma^2_{y_i} = \frac{N}{2^l} \sigma^2_{y_j} \tag{23}$$

where (i) is the set of indices of the columns satisfying the condition of Lemma 4 and

$$j = \frac{kN_1}{2^l} + 1$$

Theorem 2. If the correlation matrix of the vector X is such that

$$r_{ij} = r_{|i-j|} = r(\tau) = 1 - k\tau, \quad k\tau \le 1, \quad \tau = 1, \ldots, N \tag{24}$$

A_N is a normialized canonical Hadamard matrix, and $Y_N = X \cdot A_N$, then

1) The coordinates y_i of the vector Y can, for $i > 1$, be partitioned into $\log_2 N$ groups such that each group contains coordinates with equal variances and the dimensions of the groups are $N/2^{b^i}$, where b_i is determined from (16). Here, $\sigma^2_{y_i} > \sigma^2_{y_j}$ if $b_i > b_j$.

2) The least mean square error σ^2_{Nn}, determined by (3), is independent of N for fixed ratio n/N and is equal to

$$\sigma^2_{Nn} = 1 - \frac{1}{N} \sum_{i=1}^{n_0} \sigma^2_{y_i} \tag{25}$$

where $N_0 = N/2^S$ (S being is the largest integer such that $n = 2^S n_0$ for odd n_0) and $\delta^2_{y_{i0}}$ is the variance in the coodinates of the vector $Y_{N_O} = XA_{N_O}$.

If the correlation function r(t) of the process x(t) has a nonzero derivative at the point (+0), then r(t) can be represented in the form

$$r(t) = 1 + r'(0)t + \frac{r''(\xi)}{2} t^2 \qquad 0 < t < T \tag{26}$$ /525

where $0 < \xi < T$. One can show that, if N/n = k.2 and

$$T \left| \max_{0 \leq t \leq T} r''(t) \right| < \frac{7\, r'(0)\, k^2}{6 N^2} \tag{27}$$

then δ^2_{Nn} is independent of N and is equal to

$$\delta^2_{Nn} = \left(1 - \frac{1}{N_0}\right) \delta^2_{y_{10}} \tag{28}$$

where $x^2_{y_{10}}$ is the variance of the first coordinate of the vector Y_{N_0} and $N_O = k$.

If the correlation function r(t) has a first derivative equal to 0 at the point 0, it can be represented in the form

$$r(t) = 1 + \frac{r''(0)}{2} t^2 + \frac{r'''(\xi)}{6} t^3 \qquad t < T \tag{29}$$

By using the expression (14) of Lemma 3, we can show that, if

$$\left(\frac{N+1}{N}\right)^5 \cdot \frac{T^3 (1 + \log N)}{60} \left| \max_{0 \leq t \leq T} r'''(t) \right| < \delta_0^2 \tag{30}$$

where δ_0^2 is the admissible mean square error, then

$$\frac{n}{N} \leq \frac{1 + \log N}{N} \tag{31}$$

We conclude from (27)-(30) that the use of Hadamard filters is most effective for processes with correlation function close to quadratic and is less effective for processes with correlation function close to linear.

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